SOLUTIONS TO SELECTED QUESTIONS IN HOMEWORK 12

MATH 241

12.1.6

Proof. By our formula on the indefinite integral of $e^x \sin x$, $< f_1, f_2 >= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} e^x \sin x dx = e^x(\frac{\sin x - \cos x}{2})|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} = e^{\frac{5\pi}{4}}(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}) - e^{\frac{\pi}{4}}(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}) = 0$. So e^x and $\sin x$ are orthogonal on $[-\frac{\pi}{4}, \frac{5\pi}{4}]$. □

12.1.18

Proof. $< x + c_1x^2 + c_2x^3, x >=< x, x > +c_1 < x^2, x > +c_2 < x^3, x >$, and $< x, x >= \int_{-2}^{2} x^2 dx = \frac{1}{3}(8 - (-8)) = \frac{16}{3}$, $< x^2, x >= 0$ since $x^2 \cdot x = x^3$ is an odd function, its integral on $[-2, 2]$ must be zero, and $< x^3, x >= \int_{-2}^{2} x^4 dx = \frac{1}{5}(32 - (-32)) = \frac{64}{5}$. So $< x + c_1x^2 + c_2x^3, x >= \frac{16}{3} + \frac{64}{5}c_2$. To make $x + c_1x^2 + c_2x^3$ orthogonal to x, we want this to be zero, so $c_2 = -\frac{\frac{16}{3}}{\frac{64}{5}} = -\frac{5}{12}$.

$< x + c_1x^2 + c_2x^3, x^2 >=< x, x^2 > +c_1 < x^2, x^2 > +c_2 < x^3, x^2 >$. $< x, x^2 >=< x^3, x^2 >= 0$ since their product functions are odd functions, and $< x^2, x^2 >= \int_{-2}^{2} x^4 dx = \frac{64}{5}$. So $< x + c_1x^2 + c_2x^3, x^2 >= \frac{64}{5}c_1$. To make $x + c_1x^2 + c_2x^3$ orthogonal to x^2, we want this to be zero, so $c_1 = 0$. □

12.1.21

Proof. (b) $\sin x$'s fundamental period is 2π, so $\sin \frac{4}{L}x = \sin(\frac{4}{L}x + 2\pi) = \sin \frac{4}{L}(x + \frac{L}{2}\pi)$. So the fundamental period is $\frac{L}{2}\pi$. In general, $\sin \omega x$ or $\cos \omega x$'s fundamental perido is $\frac{2\pi}{\omega}$.

(e) $\sin 3x$ has fundamental period $\frac{2}{3}\pi$, $\cos 2x$ has fundamental period π, so their sum should have fundamental period as the least common multiple of $\frac{2}{3}\pi$ and π. If we assume this number is T, then there should be two integers m, n such that $T = m \cdot \frac{2}{3}\pi = n \cdot \pi$. This yields $\frac{m}{n} = \frac{3}{2}$. The smallest positive integers m, n that satisfy this is $m = 3, n = 2$, therefore $T = 2\pi$ is the fundamental period of $\sin 3x + \cos 2x$. □

12.2.13

Proof.

$$a_0 = \frac{1}{5}[\int_{-5}^{0} dx + \int_{0}^{5} (1 + x)dx] = \frac{1}{5}[5 + (x + \frac{x^2}{2})|_0^5] = \frac{1}{5}[5 + 5 + \frac{25}{2}] = 2 + \frac{5}{2} = \frac{9}{2}$$

$$a_n = \frac{1}{5}[\int_{-5}^{0} \cos \frac{n\pi}{5}xdx + \int_{0}^{5} (1 + x)\cos \frac{n\pi}{5}xdx] = \frac{1}{5}[\int_{-5}^{0} \cos \frac{n\pi}{5}xdx + \int_{0}^{5} \cos \frac{n\pi}{5}xdx + \int_{0}^{5} x\cos \frac{n\pi}{5}xdx]$$

$$= \frac{1}{5}[\int_{-5}^{5} \cos \frac{n\pi}{5}xdx + \int_{0}^{5} x\cos \frac{n\pi}{5}xdx]$$

The first term is zero by period argument, and we calculate the second term by integration by parts:

$$\frac{1}{5}\int_0^5 x\cos\frac{n\pi}{5}xdx = \frac{1}{5}\int_0^5 xd(\frac{5}{n\pi}\sin\frac{n\pi}{5}x) = \frac{1}{n\pi}\int_0^5 xd(\sin\frac{n\pi}{5}x)$$

$$= \frac{1}{n\pi}[(x\sin\frac{n\pi}{5}x)|_0^5 - \int_0^5 \sin\frac{n\pi}{5}xdx] = -\frac{1}{n\pi}\int_0^5 \sin\frac{n\pi}{5}xdx = \frac{1}{n\pi}\cdot\frac{5}{n\pi}\cos\frac{n\pi}{5}x|_0^5 = \frac{5}{n^2\pi^2}(\cos n\pi - 1) = \frac{5((-1)^n-1)}{n^2\pi^2}$$

Similarly,

$$b_n = \frac{1}{5}[\int_{-5}^0 \sin\frac{n\pi}{5}xdx + \int_0^5 (1+x)\sin\frac{n\pi}{5}xdx] = \frac{1}{5}[\int_{-5}^5 \sin\frac{n\pi}{5}xdx + \int_0^5 x\sin\frac{n\pi}{5}xdx]$$

The first term is zero by period argument, and we calculate the second term by integration by parts:

$$\frac{1}{5}\int_0^5 x\sin\frac{n\pi}{5}xdx = \frac{1}{5}\int_0^5 xd(-\frac{5}{n\pi}\cos\frac{n\pi}{5}x) = -\frac{1}{n\pi}\int_0^5 xd(\cos\frac{n\pi}{5}x)$$

$$= -\frac{1}{n\pi}[(x\cos\frac{n\pi}{5}x)|_0^5 - \int_0^5 \cos\frac{n\pi}{5}xdx] = -\frac{1}{n\pi}[5\cos n\pi - \frac{5}{n\pi}\sin\frac{n\pi}{5}x|_0^5] = -\frac{5(-1)^n}{n\pi}$$

Therefore the Fourier series for $f(x)$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi}{5}x + b_n\sin\frac{n\pi}{5}x = \frac{9}{4} + \sum_{n=1}^{\infty}\frac{5((-1)^n-1)}{n^2\pi^2}\cos\frac{n\pi}{5}x - \frac{5(-1)^n}{n\pi}\sin\frac{n\pi}{5}x$$

□

Fall 11, 9

Proof.

$$a_0 = \frac{1}{\pi}\int_0^{\pi} dx = 1$$

$$a_n = \frac{1}{\pi}\int_0^{\pi}\cos nxdx = \frac{1}{n\pi}\sin nx|_0^{\pi} = 0$$

so all the a_k's are zero, therefore $a_0 + \sum_{k=1}^{\infty} a_k^2 b_k = a_0 = 1$. □